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Project **Insurance Contracts**

Topic **Option pricing and risk**

Purpose of this paper

1. The purpose of this paper is to explain the role that risk adjustments play in standard option pricing techniques.

2. All standard option pricing models include such adjustments, but they are usually present in a form that makes it easy to overlook them. Understanding this fact should make it easier to understand the rationale for risk adjustments in the measurement model being developed for insurance contracts. It may also provide some intuitions that may help in comparing and evaluating various techniques for estimating such risk adjustments.

3. The paper does not present recommendations. We intend to walk through the paper during the joint Board meeting as background for the discussion on risk adjustments.

Structure of the paper

4. The rest of this paper is divided into the following sections:

   (a) Risk adjustments in lattice models (paragraphs 5-21)

   (b) Risk adjustments in the Black-Scholes model (paragraphs 22-28)

   (c) Incomplete markets (paragraphs 29-31)

(d) Option pricing as a guide to evaluating models for risk adjustments (paragraphs 32-35)

(e) Summary (paragraphs 36-39)

Risk adjustments in lattice models

5. The easiest way to explain how option pricing models deal with risk is to walk through a simple example. Assume the following fact pattern. The objective is to price an option at time T0 (ie T = 0) with the following features:

    (a) Option to buy 1 share of company A

    (b) Expiry date: time T2. Early exercise is not permitted.

    (c) Strike price: CU95

    (d) Current share price: CU100

    (e) At time T1, the share price may increase by 10% (to CU110) with probability 80% or decrease by 9.09% (to CU90.91) with probability 20%.

    (f) Similarly, at time T2, the share price may increase by 10% (from CU110 to CU121 or from CU90.91 back up to CU100) with probability 80% or decrease by 9.09% (from CU110 back down to CU100 from CU90.91 toCU82.64) with probability 20%.

    (g) Investors can borrow or invest unlimited amounts of money at the risk-free rate of 5%.

6. We now discuss how to value this option using a binomial tree (lattice).

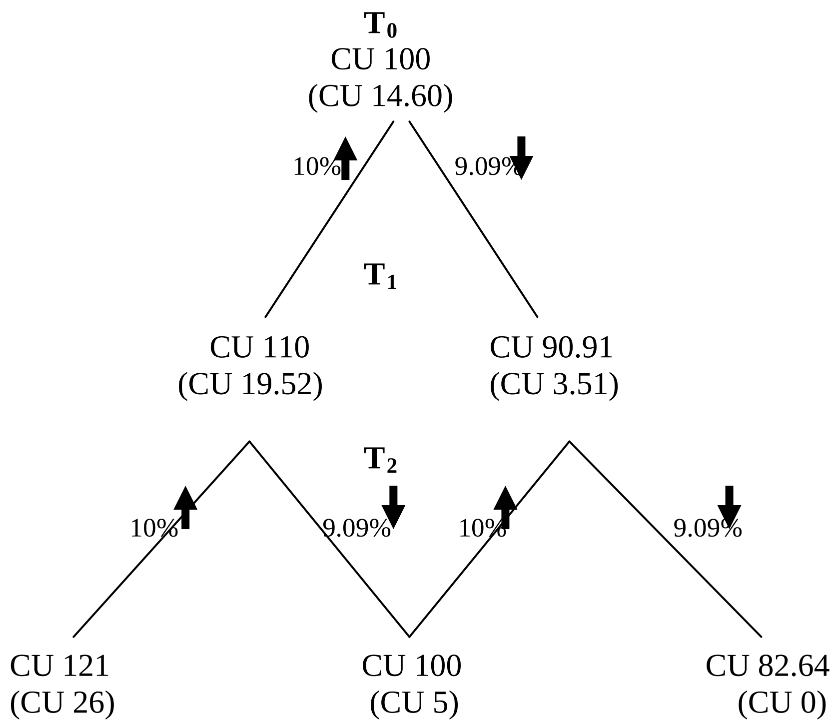
    (a) Consider first what happens if at time T1 the share price is CU110. Instead of buying the option, an investor could buy a replicating portfolio comprising one share (fair value CU110), financed by a loan of CU90.48. The total value of this package is CU19.52 (CU110 – CU90.48). At time T2, the value of this package will be either CU26 (CU121 – CU95) if the share price is CU121, or CU 5 (CU100 – CU95) if the share price is CU100. These payoffs are the same as the payoffs from the option. Therefore, unless an arbitrage possibility exists, when the share price is CU110 at T1, the value of the option at that time must be CU19.52.

    (b) Consider now what happens if at time T1 the share price is CU90.91. Instead of buying the option, an investor could buy a replicating portfolio comprising 0.2881 shares (fair value CU26.19), financed by a loan of CU22.68. The total value of this package is CU3.51 (CU26.19 – CU22.68). At time T2, the value of this package will be either CU5 (CU28.81 – CU23.81) if the share price is CU100, or CU0 (CU23.81 – CU23.81) if the share price is CU82.64. Again, these payoffs are the same as the payoffs from the option. Therefore, unless an arbitrage possibility exists, when the share price is CU90.91 at T1, the value of the option at that time must be CU3.51.

(c) Finally, we consider what happens at T0. From (a) and (b) the value of the option will either CU19.52 (if the share price at T1 is CU110) or CU3.51 (if the share price at T1 is CU90.91). Suppose an investor buys 0.8386 shares (value CU83.86) and borrows CU69.26. The value of this replicating portfolio is CU14.60 at T0. At T1, its value is either CU19.52 (CU92.24 - CU72.72) if the share price is CU110 or CU3.51 (CU76.23- CU72.72) if the share price is CU90.91. Once more, the value of the replicating portfolio at T1 is the same as the value of the option. Therefore, unless an arbitrage is possible, the value of the option must be CU14.60.

7. On the following page, we summarise this information on a binomial tree (lattice). The tree shows for each date the possible share price, the corresponding option price and the possible increase or decrease in share price during the next time period.

Simple Example – Binomial tree.



At $\mathbf{T_0}$ the share price is CU 100; at $\mathbf{T_1}$ share price may rise up to CU 110 or decrease at CU 90.91; at $\mathbf{T_2}$ share price can assume the following values CU 121, CU 100 or CU 82.64.

For each scenario, just below share prices, option values are shown in (parentheses).

Each scenario is built assuming an increase ( ) of 10% or decrease ( ) of 9.09% in the share price.

The tree does not show the **probability** of an increase or decrease in share price at each date because that information is not needed in determining the option prices.

8. The appendix to this paper presents the above data in a tabular format.

9. The above example is simplistic, but it does illustrate important points that are common to all option pricing models:

   (a) The valuation rests largely (and in this case entirely) on finding a replicating portfolio that produces the same cash flows as the item being valued. In this case, the item being valued is an option and the replicating portfolio is made up of shares and borrowings.

(b) If the replicating portfolio produces the same cash flow as the option in all outcomes, the option must have the same value as the replicating portfolio (presuming that arbitrage is not possible).

(c) Strikingly, the model does not include any explicit inputs to reflect the extent to which investors are averse to risk (ie the degree of their risk aversion). Nor does it explicit estimates of probabilities. However, both these items are there implicitly but buried in the inner workings of the valuation model. In the following sections, we look at other models that build on the same conceptual foundations and that use risk adjustments more explicitly.

A form that will be useful in more realistic examples

10. We now recast the above valuation in another form. This will illustrate a technique not needed for this particular example, but that technique is often the most practical approach in more realistic examples.

11. For simplicity, we will focus on the case when the share price has reached CU110 at T1. The possible payouts are CU121 and CU100. Discounting at the risk-free rate of 5%, those two outcomes have present values at T1 of CU115.24 and CU95.24 respectively. Now, it is possible to value the option at T1 by applying "probabilities" to the present value of the outcomes. The following table summarises the only probabilities that will produce the option value of CU19.52 already derived in paragraph 6(a).

Share price	Cash flow CU	Discounted 5%	Probability	Weighted
121	26.00	24.76	0.7381	18.28
100	5.00	4.76	0.2619	1.25
Total			1.000	19.52

12. The "probabilities" used in the above table are not real estimates of the likelihood that each outcome will occur. Instead, they are adjusted for risk, and

so we will call them risk-adjusted probabilities. These risk-adjusted "probabilities" reflect a blend of two factors:

(a) Estimates by market participants of the "true" likelihood of each outcome

(b) Market participants' preferences for payouts in some scenarios over the same amount of payout in other scenarios.

13. It is instructive to compare the above table with the following table prepared using the "true" probabilities of each outcome. These true probabilities are an estimate of the likelihood of each outcome. In other words, if an entity makes accurate estimates of the true probabilities for many events, the actual frequency of each outcome will tend to be close to the estimate of the "true" probability. Whereas the "risk-adjusted" probabilities can be calculated from other data given in this example, the "true" probabilities cannot be inferred in this way. Instead, they must be estimated separately. In this example, the "true" probabilities are stated in paragraph 5(e), as part of the fact pattern assumed for this example.

Share price	Cash flow CU	Discounted 5%	True Probability	Weighted
121	26.00	24.76	0.8000	19.81
100	5.00	4.76	0.2000	0.95
Total			1.000	20.76

14. Using the "true" probabilities, the value of the option is CU20.76, whereas using the "probabilities" used in the first table, its value is only CU19.52 (consistent with the value derived earlier in this paper). The difference of CU1.24 (CU20.76-CU19.52) is, in effect, the premium that market participants require for bearing the risk inherent in the option. That is, market participants put a higher weighting (26.19%) on the unfavourable outcome (a share price of CU100 at T2) than on its "true" probability (20.00%).

15. To re-express this in the terms we have been using for insurance contracts, the sum of the first two building blocks (cash flows and time value of money) is CU20.76. The risk adjustment (third building block) is CU1.24, resulting in an option price of CU19.52. If we did not include a risk adjustment, we would produce a measurement that is conceptually inconsistent with option pricing models. [Note that a risk-adjustment **decreases** the **asset** value of the option in this example, whereas a risk-adjustment **increases** the **liability** value of the first two building blocks for insurance contacts. This is because a less favourable outcome gives **lower** net cash inflows for the option, but for insurance contracts a less favourable outcome leads to **higher** net cash outflows.]

A note on terminology

16. Unfortunately, the terminology used in this area is very confusing. The term most commonly used to describe the risk-adjusted probabilities described in paragraph 12 is "risk-neutral" probabilities. Thus, contrary to appearances, the term "risk-neutral" probability actually signals that the measurement includes (rather than excludes) a risk adjustment. To minimise confusion, the rest of this paper uses the term "risk-adjusted" to describe these probabilities.

17. The term most commonly used to describe an estimate of the likelihood of occurrence, unadjusted for risk, is "real-world" probability.

Risk adjusted probabilities in lattice and binomial models

18. In the simple example presented in paragraphs 5-9 above, we did not need to use probabilities to price the options, because the simplest approach was to determine the composition of the replicating portfolio and then observe the price of that portfolio in the market.

19. However, for many more realistic examples, the simplest and most robust way to value a derivative is to use a lattice (or a binomial model) to generate the expected present value of the outcomes, determined using **risk-adjusted** ("risk-neutral") probabilities, discounted at the risk free rate.

20. Thus, these lattice and binomial models **incorporate a risk adjustment.**

21. The risk adjustment is difficult to see because it is derived by using risk-adjusted probabilities.

Risk adjustments in the Black-Scholes model

22. Let us now consider these ideas in the context of the Black-Scholes model for pricing options. The Black-Scholes formula for the price of a European call option is as follows:

> Call option price $= SN(d_1) – Xe^{-rt}N(d_2)$
>
> where $d_1 = [\ln (S/X) + (r + ½\sigma^2)t] / \sigma\sqrt{t}$
>
> $d_2 = d_1 - \sigma\sqrt{t}$
>
> r is the risk-free rate
>
> The other symbols used in this formula are not important for the purpose of this paper.

23. We are not going to explain this formula in detail because our objective is simply to explain where this formula incorporates risk. We instead summarise the meaning of the two main components of this formula. We then note briefly where the risk adjustment comes into this formula. We do not provide detailed justifications for the statements that follow because that would take pages of mathematical equations.

24. The formula has two parts:

    (a) The left hand part $[SN(d_1)]$ is the expected present value of the higher of (i) the share price at the expiration date of the option and (ii) the strike price.

    (b) The right hand part $[Xe^{-rt}N(d_2)]$ is the expected present value of the strike price. (Note that if the option is out of the money at expiry, (a)-(b) = zero and so the formula attributes zero value to those scenarios.)

25. In essence, then, the formula prices the option by considering the price of a replicating portfolio made of the underlying shares, financed by borrowings at the risk free rate.

26. There are two critical things to note about this formula:

    (a) Paragraph 23 refers to expected present values. These are determined using **risk-adjusted** (ie risk-neutral) probabilities; they are **not** determined using real-world probabilities.

    (b) The symbol d_1 in the formula includes the term r (ie the risk-free rate). This is because when risk-adjusted probabilities are used, the expected return on the shares becomes equal to the risk-free rate. In contrast, if "real-world" probabilities were used, the formula would need to be modified to replace the risk-free rate by the "real" expected return on the shares.

27. Thus the Black-Scholes formula **incorporates a risk adjustment.**

28. The risk adjustment is difficult to see because it is derived by using risk-adjusted probabilities.

Incomplete markets

29. Finance theorists distinguish complete and incomplete markets. A complete market is one in which a replicating portfolio can be found for the instruments being valued. In a complete market, every instrument can be priced unambiguously using the replicating portfolio.

30. In an incomplete market, a replicating portfolio does not exist for every instrument. This means that there is not a unique price for every instrument. However, it is possible to place some bounds around the prices for those instruments. Insurance markets are, of course, incomplete.

31. It is worth remembering that Black-Scholes is also generally applied in incomplete markets. Most of the inputs used in the Black-Scholes formula are observable but one input is not observable. This is the volatility (σ). Volatility refers here not to actual volatility in the past but to estimated volatility over the life of the option. It is possible to estimate the market's view of volatility ("implied volatility") by reference to the prices of other traded instruments that depend on volatility. However, estimates of implied volatility rely on the application of models to observable prices, and no model is perfect. Thus, there will generally be a range of reasonable estimates of implied volatility.

Option pricing as a guide to evaluating models for risk adjustments

32. A key notion underlies much of this paper: risk-averse market participants put more weight on some outcomes than on other outcomes. That differential weighting finds expression in the use of risk-adjusted probabilities.

33. The notion that different weightings apply to different outcomes may help in evaluating the various techniques we might consider for estimating risk adjustments. In other words, how does a particular technique weight different outcomes?

34. To take one example, consider Value at risk (VaR) and conditional tail expectation (CTE). For ease of comparison, we will contrast VaR at the 95% level with CTE at the 95%.

    (a) VaR ranks all the outcomes and then place 100% weighting on the single outcome at 95%, and zero weighting on all others.

    (b) CTE ranks outcomes and then places zero weight on outcomes from 0%-95% and equal weight on all outcomes from 95%-100%.

    (c) VaR is probably easier to implement and probably somewhat easier to explain to less sophisticated users. On the other hand, CTE captures more information about the distribution. Indeed, unlike VaR, it distinguishes between probability distributions that look very different out in the tails between 95% and 100%.

35. Other factors to be considered in evaluating techniques include ease of implementation, understandability, ease of benchmarking, ability to generate simple disclosures.

Summary

36. Although people sometimes argue that option pricing models do not include risk, this is not true. All mainstream option pricing models, including Black-Scholes and lattice models, involve adjustments of this kind because all of them build on the notions (no arbitrage and replicating portfolio) that underlie “risk-neutral” models. Essentially, all these models rely on the creation of an actual

or synthetic replicating portfolio that generates the same cash flows as the asset being valued.

37. Unfortunately, it is not always easy to see where the risk-adjustment is included: it can be buried quite deep in the workings of the model; moreover, because practitioners take it for granted, its presence may be signalled very unobtrusively. Among the terms that indicate the presence of a risk adjustment are risk-neutral(!), deflator, pricing kernel, no arbitrage, arbitrage-free, stochastic discount factor, martingale, Q probability measure (as opposed to P probability measure).

38. Including a risk adjustment in the measurement of an insurance contract is consistent with the fact that option pricing models also include risk adjustments. Of course, the more difficult issue is determining what techniques to use in particular circumstances.

39. Unobservable inputs derived using models are needed both in many practical applications of option pricing models and in determining risk adjustments for insurance contracts. The range of reasonably supportable amounts for those inputs depends on the circumstances.

	Lattrice--simple example												
(a)	Option to buy x share(s) of company A				1								
(b)	Expiry date T2: early exercise not permitted												
(c)	Strike price at T0:			CU	95								
(d)	Share price at T0:			CU	100								
(e)	Investors can borrow or invest unlimited amounts of money at the risk free rate of:								5.00%				
(f)	The share price may increase at:			T1	from	100	by	10.00%	to	110.00	probability	80%	
		or decrease by		T1	from	100	by	9.09%	to	90.91	probability	20%	
(g)	The share price may increase at:			T2	from	110.00	by	10.00%	to	121.00	probability	80%	
				T2	from	90.91	by	10.00%	to	100.00	probability	80%	
		or decrease by		T2	from	110.00	by	9.09%	to	100.00	probability	20%	
				T2	from	90.91	by	9.09%	to	82.65	probability	20%	
	Lattice												
				T0			T1			T2	or	T2	
	Investor		buys shares	1.0000	at	110.0000	110.00		Share price	121.00		100.00	
	Investor		borrows discounted strike price CU				(90.48)		Strike price	(95.00)		(95.00)	
			Value (net)				19.52			26.00		5.00	
											less than strike price		
									Share price	100.00		82.65	
	Investor		buys shares	0.2881	at	90.9100	26.19		Frac share	28.81		23.81	
	Investor		borrows discounted floor				(22.68)			(23.81)		(23.81)	
			Value (net)				3.51			5.00		-	
											total can't be less than 0.		
							92.25						
							(72.72)	(79.67)	strike price				
							19.52						
	Investor	buys shares	0.8386	83.86									
	Investor	borrows CU		(69.26)									
		Value (net)		14.60									
							76.24						
							(72.72)	(79.67)	strike price				
							3.51						